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SOLUTION OF PROBLEMS IN NUMBER SIX VOL. FIVE.

(*Continued from page* 30.)

SOLUTION OF 237 BY CHAS. H. KUMMELL, DETROIT, MICHIGAN.

We have

$$u=\int_0^{\pi}\left[\frac{\sin(n—\frac{1}{2})(\theta+\pi)}{\sin\frac{1}{2}(\theta+\pi)}+1\right]d\theta = u'+\tfrac{1}{2}\pi. \tag{1}$$

Placing $\theta = 2\varphi$ we have

$$u'=\int_0^{\frac{1}{2}\pi}\frac{\sin(2n-1)(\frac{1}{2}\pi+\varphi)}{\sin(\frac{1}{2}\pi+\theta)}. \tag{2}$$

We have $\sin nx$ = real part of $(\cos x+i\sin x)^n\div i$

$$=\sin x\left[n\cos^{n-1}x—\frac{n(n—1)(n—3)}{1.2.3}\cos^{n-3}x\sin^2x+\ldots.\right].$$

Applying this formula we have in (2)

$$u'=\int_0^{\frac{1}{2}\pi}d\varphi\left[(2n—1)\sin^{2n-2}\varphi—\frac{2n—1}{1}.\frac{2n—2}{2}.\frac{2n-3}{3}\sin^{2n-4}\varphi\cos^2\varphi\right.$$
$$\left.+\frac{2n—1}{1}\ldots\frac{2n—5}{3}\sin^{2n-6}\cos^4\varphi—\ldots.\right]. \tag{3}$$

We have

$$\int_0^{\frac{1}{2}\pi}d\varphi\sin^{p-1}\varphi\cos^{q-1}\varphi=\frac{1}{2}.\frac{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}{\Gamma(\frac{p+q}{2})}. \tag{4}$$

Applying this formula term by term to (3) we obtain

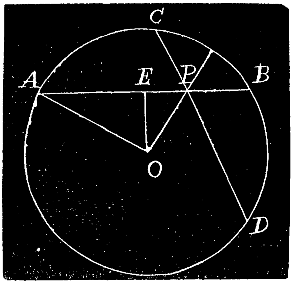
$$u'=(n—\tfrac{1}{2})\left[\frac{\Gamma(n-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(n)}—\frac{n-1}{1}.\frac{2n—3}{3}.\frac{\Gamma(n-\frac{3}{2})\Gamma(\frac{3}{2})}{\Gamma(n)}\right.$$
$$\left.+\frac{n—1}{1}.\frac{2n—3}{3}.\frac{n—2}{2}.\frac{2n—5}{5}.\frac{\Gamma(n-\frac{5}{2})\Gamma(\frac{5}{2})}{\Gamma(n)}—\ldots..\right]$$
$$=\frac{\Gamma(n+\frac{1}{2})\pi^{\frac{1}{2}}}{\Gamma(n)}\left[1—\frac{n—1}{1}.\frac{1}{3}+\frac{n—1}{1}.\frac{n—2}{2}.\frac{1}{5}—\ldots..\right]$$
$$=\frac{\Gamma(n+\frac{1}{2})\pi^{\frac{1}{2}}}{\Gamma(n)}\int_0^1 dx\left[1—\frac{n—1}{1}x^2+\frac{n—1}{1}.\frac{n—2}{2}x^4—..\right]$$
$$=\frac{\Gamma(n+\frac{1}{2})\pi^{\frac{1}{2}}}{\Gamma(n)}\int_0^1 dx(1—x^2)^{n-1}$$
$$=\frac{\Gamma(n+\frac{1}{2})\pi^{\frac{1}{2}}}{\Gamma(n)}\int_0^{\frac{1}{2}\pi}d\varphi\cos^{2n-1}\varphi\quad(x=\sin\varphi)$$
$$=\frac{\Gamma(n+\frac{1}{2})\pi^{\frac{1}{2}}}{\Gamma(n)}\times\frac{1}{2}\frac{\Gamma(\frac{1}{2})\Gamma(n)}{\Gamma(n+\frac{1}{2})}\quad\text{[by (4)]}$$
$$=\tfrac{1}{2}\pi.$$

We have then, by (1), $u = u'+\frac{1}{2}\pi = \pi$.

238. "Two points are taken at random in the surface of a circle, radius r, one of them being confined to a given radius, and a chord drawn through the points. Find (1) the chance that the length of the chord does not exceed $2a$, and (2) the chance that it does not exceed the radius of the circle."

SOLUTION BY ARTEMAS MARTIN, M. A., EDITOR MATH. VISITOR, ERIE, PA.

1. Let O be the center of the circle, and P one of the random points. Through P draw two chords, length $2a$. If the other point is taken in the surface APC, or in the surface BPD, the chord drawn through them will not exceed the length $2a$.



Let $OP = x$, $S =$ surface $APC =$ surface BPD, $m = \surd(r^2 - a^2) = OE$, and $p_1 =$ the chance required.

We readily find $S = r^2 \cos^{-1}(m \div x) - m(x^2 - m^2)^{\frac{1}{2}}$.

Hence $$p_1 = \int_m^r 2S dx \div \int_0^r \pi r^2 dx,$$

$$= \frac{2}{\pi r^3} \int_m^r \left[r^2 \cos^{-1}\left(\frac{m}{x}\right) - m\surd(x^2 - m^2) \right] dx,$$

$$= \frac{1}{\pi}\left[2\cos^{-1}\left(\frac{m}{r}\right) - \frac{m\surd(r^2 - m^2)}{r^2} - \frac{m(2r^2 - m^2)}{r^3} \log\left(\frac{r + \surd(r^2 - m^2)}{m}\right) \right],$$

$$= \frac{1}{\pi}\left[2\sin^{-1}\left(\frac{a}{r}\right) - \frac{a\surd(r^2 - a^2)}{r^2} - \frac{(r^2 + a^2)\surd(r^2 - a^2)}{2r^3} \log\left(\frac{r+a}{r-a}\right) \right].$$

2. Taking $a = \frac{1}{2}r$, we have

$$p_2 = \frac{1}{3} - \frac{1}{\pi}\left(\frac{\surd 3}{4} - \frac{5\surd 3}{16} \log 3 \right).$$

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239. "A can plant thirty-six per cent of his arrows within a circular target ten inches in diameter at the distance of one hundred yards; B can plant sixty-four per cent of his arrows within a circle thirteen and one third inches in diameter at the same distance.—Prove that B's skill is greater than A's."

SOLUTION BY CHAS. H. KUMMELL.

Assuming that the marks made in shooting the arrows are distributed according to the laws of probability, we have, if h_1 is the skill of A and h_2 that of B, and $\varDelta$ any given distance of a mark from the centre:

$$\frac{h_1}{\sqrt{\pi}}\int_{-10}^{10} d\varDelta e^{-h_1^2\Delta^2} = 0.36 = \frac{2h_1}{\sqrt{\pi}}\int_{0}^{10} d\varDelta e^{-h_1^2\Delta^2};\ \frac{2h_2}{\sqrt{\pi}}\int_{0}^{13\frac{1}{3}} d\varDelta e^{-h_2^2\Delta^2} = 0.64.$$

Place $h_1\varDelta = \theta$ and also $h_2\varDelta = \theta$, that is, consider θ the variable of the integration instead of $\varDelta$, then we have

$$\frac{2}{\sqrt{\pi}}\int_{0}^{h_1 10} d\theta e^{-\theta^2} = 0.36; \qquad \frac{2}{\sqrt{\pi}}\int_{0}^{h_2 13\frac{1}{3}} d\theta e^{-\theta^2} = 0.64.$$

There are tables of these integrals, for instance, Table IX in Chauvenet's Method of Least Squares. From this we find

$$h_1 10 = 0.33071, \therefore h_1 = 0.033071;$$
$$h_2 13\tfrac{1}{3} = 0.64727, \therefore h_2 = 0.048545.$$

Therefore $h_2 = 1.4679h_1$, or B shoots 1.4679 times better than A.

SOLUTION BY R. J. ADCOCK, MONMOUTH, ILLINOIS.

Since A can plant $\frac{36}{100}$ of his arrows within 5 inches of the centre, the probability that he can shoot the whole number, n, within that distance is $\frac{9}{25}$. Hence in eq'n (4), page 21, Vol. V, where $l = \sqrt{[pS(d_1^2) \div n]}$, making $l = 5$, and $p = \frac{9}{25}$, there results

$$5 = \frac{3}{5}\sqrt{\left[\frac{S(d_1^2)}{n}\right]} \text{ or } \sqrt{\left[\frac{S(d_1^2)}{2n}\right]} = \frac{25}{3\sqrt{2}} = \frac{25\sqrt{2}}{6} \text{ inches},$$

which, by eq'n (6) on the same page (where the sign of equality is wanting after l through a mistake in printing), is the probable error of a shot by A.

In the same way we find that the probable error of one of B's shots is $(25\sqrt{2}) \div 6$. Therefore they must shoot with an equal degree of accuracy.

SOLUTION BY PROF. P. E. CHASE.

The diameters of targets in which there would be an equal chance to hit or miss, are

$$\text{for A, } 10\sqrt{\left[-\frac{\log 2}{\log(1-.36)}\right]} = 12.46;$$

$$\text{for B, } \frac{40}{3}\sqrt{\left[-\frac{\log 2}{\log(1-.64)}\right]} = 10.98.$$

B's skill is therefore $(12.46 \div 10.98) = 1.13 \times$ A's.

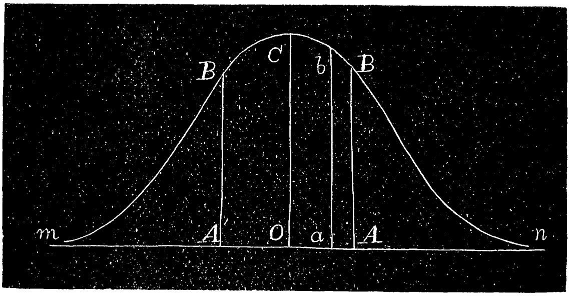
The above is based on Herschel's formula

$$M = (\tfrac{1}{2})^{\frac{r^2}{a^2}},$$

in which M = probability of missing, in any single shot, a circular area of radius r at whose centre the shot is aimed; a = radius of circular area within which his skill would, on the average of an infinite number of shots, enable him to plant half the total number discharged. (Familiar Lectures on Scientific Subjects. London and New York, 1867, p. 495.)

[If O is the origin and x, y rectangular coordinates, the equation

$$y = ce^{-h^2x^2}$$



will represent the curve $mB'CBn$, which is called the *probability curve*; and, assuming the whole area $= 1$, if the area $OABC = OA'B'C = \frac{1}{4}$, OA is call'd the *probable error*.

In the case under consideration, suppose O to represent the center of the target at which the marksman aims, and let Oa denote the radius of the target, then the probability of hitting the target is $2OabC$; and, for the marksman A, there are give $Oa = 5$ inches and area $2OabC$ (= probability of hitting the target) = 0.36; and for the marksman B, $Oa = 6\frac{2}{3}$ inches and area $2OabC = 0.64$. But the area $2OabC$ is represented by the int'gl

$$\frac{2}{\sqrt{\pi}}\int_0^{hx} e^{-h^2x^2}\, d.hx,$$

in which h is the precision of the marksman and is $= 0.4769 \div r$, r being the probable error OA; and in the table referred to by Mr. Kummell in the foregoing solution, the values of hx are calculated for the different values of P. Taking from the table the values corresponding to $P_1 = 0.36$, and $P_2 = 0.64$, and dividing respectively by 5 and $6\frac{2}{3}$, we find the precision of A and B as determined by Mr. Kummell, and by Professor Kershner whose solution is substantially the same as Mr. Kummell's.

The first of the foregoing solutions is therefore correct if the hypothesis upon which it is based is correct; i. e., if the equation

$$y = ce^{-h^2x^2} \tag{1}$$

represents the relation between an error and its probability.

In the second solution, given above, the result obtained is at variance with the first. It will be seen, however, that the hypothesis upon which Mr. Adcock's solution is based virtually assumes that all possible errors are equally probable. A glance at the probability curve shows, however, that, if equation (1) represents the law of facility of error, small errors occur more frequently than large ones. Moreover *our* experience, and probably the experience of most persons, is confirmatory of that fact.

That Mr. Adcock assumes all possible errors equally probable, appears not only from his discussion at pages 21 and 22 of Vol. V, upon which the above solution is based, but is also deducible from his result here obtained.

If we suppose a circular target of radius 8⅓ inches to have 100 points uniformly distributed over its area, 36 of these points will be within a concentric circle of 5 in. radius (A's target) and 64 within a circle of 6⅔ in. radius (B's target); which proves that, if A's skill is equal to B's, great and small errors must be equally probable.

In the third solution given above the greater frequency of small than of large errors is recognized in the use of eq'n (1), but by assuming that the degree of the error varies with x [a fact included in (1)] and multiplying the differential of P by this factor a result is obtained which differs from both the others. The *reason* for multiplying the diff. of P by x is not apparent.

After the foregoing was written, we rec'd from Mr. Adcock the subjoined Revision of his paper on Limits of Errors and his solution of 239.—Ed.]

REVISION OF "LIMITS OF ERRORS", etc. (P. 21, VOL. V.)—Since "$p = n \div 4\pi m S(d_1^2) \ldots$ (1) is the probability that any point of the total number $4\pi m S(d_1^2)$ shall be at or on any point line or surface," it is, when $S(d_1^2)$ is a minimum, the probability that the point, line or surface is in its true position. For any other position of the point, line or surface, where the sum of the squares of the normals $= S(\delta^2)$, the probability is $y = n \div 4\pi m S(\delta^2)$.

Hence $$\frac{y}{p} = \frac{S(d_1^2)}{S(\delta^2)}. \tag{2}$$

Let x equal the distance between these two positions measured from the first, then by my proposition, 200 (pp. 64 and 91, Vol. V), and because that proposition may be extended to include the line and surface,

$$x^2 = [S(\delta^2) - S(d_1^2)] \div n.$$

Hence $$y = \frac{pS(d_1^2)}{S(d_1^2)+nx^2} = \frac{p}{1+[nx^2 \div S(d_1^2)]} = \frac{p}{1+c^2x^2}. \tag{3}$$

The curve represented by (3) is the true "probability curve", being that imposed by the theory of probabilities. The ordinate y is the probability, frequency or density of the number of normals for the abscissa x. The area is

$$\int_0^x y\,dx = \frac{p}{c}\tan^{-1} cx.$$

Let z equal the number of errors not greater than x, and n the number not greater than l, then

$$\frac{\tan^{-1} cx}{\tan^{-1} cl} = \frac{z}{n}, \tag{4}$$

equals the probability that an error shall not exceed x. Eq'n (4) satisfactorily agrees with the number and magnitude of errors in actual observations, as far as has been examined by me.—[Revised solution, as follows:]

$$\int_0^x y\,dx = \frac{p}{c}\tan^{-1}cx = \frac{n}{4\pi m S(d_1^2)}\sqrt{\left(\frac{S(d_1^2)}{n}\right)}\tan^{-1}\frac{x}{\surd[S(d_1^2)\div n]};$$

$$\frac{1}{4\pi m}\sqrt{\left(\frac{n}{S(d_1^2)}\right)}\tan^{-1}\frac{5}{\surd[S(d_1^2)\div n]} = \frac{9}{25},$$

$$\frac{1}{4\pi m}\sqrt{\left(\frac{n}{S(\delta^2)}\right)}\tan^{-1}\frac{\frac{1}{3}(20)}{\surd[S(\delta^2)\div n]} = \frac{16}{25};$$

$$\therefore \quad \frac{z\tan^{-1}5z}{y\tan^{-1}\frac{1}{3}(20)y} = \frac{9}{16}.$$

Hence $y > z$, and $\therefore$ B's skill is greater than A's.

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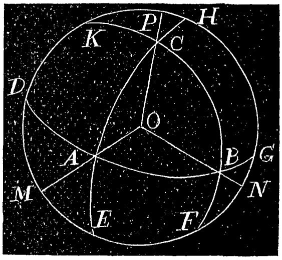
240. "If the angles θ, φ and ψ are connected by a certain relation any two of them may be the oblique angles of a spherical right triangle, and the third will be the complement of the perpendicular from the right-angle to the hypothenuse. Give a geometrical construction of the three triangles thus connected, and find the relations that exist between their sides."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.



Let ABC be a tri-rectangular triangle. With any point O in ABC as a pole describe a great circle, cutting the sides of ABC produced in D, E, F, G, H, K. Draw the quadrants OM, ON, OP through A, B, C. Then the right triangles ADE, BFG, CHK are the required triangles.

For let arc $AO = \theta$, arc $BO = \varphi$, and arc $CO = \psi$. Then since CEF is the polar triangle of AOB, we have $\angle AED = 180° - CEF = \varphi$, and $\angle BFG = 180° - CFE = \theta$. For like reasons, $\angle BGF = \psi$, $CHK = \varphi$, $CKA = \theta$, and $ADE = \psi$. We also have arc $AM = 90° - \theta$, $BN = 90° - \varphi$, and $CP = 90° - \psi$. Hence the triangles ADE, BFG and CHK have the required relation. To find the relation of the angles θ, φ and ψ, let $EAM = A$; then $\cos\varphi = \sin\theta\sin A$, and $\cos\psi = \sin\theta\cos A$, whence, by squaring, adding and reducing, we obtain

$$\cos^2\theta + \cos^2\varphi + \cos^2\psi = 1.$$

Because $DABG$, $FBCK$ and $HCAE$ are semi-circumferences, and AB, BC and CA are quadrants, we have $AD + BG = BF + CK = CH + AE$ = a quadrant, for the relation between the sides about the right angles of the triangles.

Again, since $DE = GH$, $FG = KD$ and $HK = EF$, we have $DE + FG + HK$ = a semi-circumf., for the relation between the hypothenuses.